Final Exam - Functional Analysis
M. Math I

06 May, 2022

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100.

(iii) You may directly invoke results proved in the class.

(iv) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

1. (10 points) Let $C^1[0,1]$ denote the space of C^1-functions on $[0,1]$ endowed with the sup norm inherited from $C[0,1]$. Let

$$\frac{d}{dx} : (C^1[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$$

be the differentiation operator. Prove that D is linear and has closed graph, but is not continuous.

Total for Question 1: 10

2. (15 points) A subset Y of a Hilbert space $\mathscr{H}$ is said to be an *orthonormal set* if it consists of mutually orthogonal unit vectors. If Y is an orthonormal set in $\mathscr{H}$, prove that the following conditions are equivalent:

   (i) $[Y] = \mathscr{H}$ (that is, the linear span of Y is dense in $\mathscr{H}$);

(ii) For each $u \in \mathscr{H}$, $u = \sum_{y \in Y} \langle u, y \rangle y$;

(iii) For each $u, v \in \mathscr{H}$, $\langle u, v \rangle = \sum_{y \in Y} \langle u, y \rangle \langle y, v \rangle$.

Total for Question 2: 15

3. (20 points) Let $\mathfrak{X}$ be a real Banach space and $(\mathfrak{X})_1$ denote the closed unit ball of $\mathfrak{X}$. A functional $\rho \in \mathfrak{X}^{\#}$ is said to be *norm-attaining* if there is an $x_0 \in (\mathfrak{X})_1$ such that $\rho(x_0) = \|\rho\|$. Prove that the set of norm-attaining functionals in $\mathfrak{X}^{\#}$ is norm-dense in $\mathfrak{X}^{\#}$.

   (You may assume the Ekeland variational principle stated as follows: For every $\varepsilon > 0$, there is an $x_0 \in (\mathfrak{X})_1$ such that for all $x \in (\mathfrak{X})_1$, we have

$$\rho(x - x_0) \geq -\varepsilon \|x - x_0\|.)$$

Total for Question 3: 20

4. Let X be a compact Hausdorff space and let $\mathscr{D}(X) := \{\delta_x : x \in X\} \subseteq (C(X))^{\#}$ denote the set of Dirac measures on X.

   (a) (5 points) Show that $\mathscr{D}(X)$ is the set of extreme points of the space of regular Borel probability measures on X.

   (b) (10 points) Show that $\mathscr{D}(X)$ with the weak-$*$ topology is homeomorphic to X.

   (c) (10 points) Show that every regular Borel probability measure on X is the weak-$*$ limit of finitely supported probability measures (discrete probability measures) on X.

Total for Question 4: 25

5. (15 points) Show that the closed unit ball of an infinite-dimensional Banach space is not compact (with respect to the norm topology).

Total for Question 5: 15

6. Let V be a real vector space.

   (a) (5 points) Prove that among all the convex sets B in V containing 0 as an internal point and having a given Minkowski functional p, there is a largest B_1 and a smallest B_0 with respect to set inclusion.

   (b) (10 points) Show that B_1 is the closure of B_0 in the convex core topology (the coarsest topology on V which makes all linear functionals on V continuous.)

Total for Question 6: 15